Bell polynomials in discrete probability distributions

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In this report we try to find how Bell polynomials can be used to solve the problems in the probability distribution theory. A main problem is the analysis of compound distributions and related random partitions of integer.

1 Introduction and preliminaries

Bell polynomials (in relation with statistics).

1. compound distributions
2. random partition of integer (frequency of frequencies)
3. Generalized Stirling numbers (specific subgroup of Bell polynomials)
4. moments/cumulants relationship
5. Lagrangian probability distributions

A unified way for using computer algebra systems (Wolfram Mathematica)

Compound distributions: A typical model for cluster samples.

$$Z \sim h(k), G_h(w) = \sum_{0 \le k} h(k)w^k, \quad Y \sim g(k), G_g(w), \quad k \in \mathbb{N}_0,$$

$Z_1, Z_2, \ldots$: i.i.d., Y independent

$$X = Z_1 + \cdots + Z_Y$$

"compound X by compounding Y of the compouded Z"
Gurland's notation

$$X = Y \bigvee Z, \quad (\text{or} \quad f = g \bigvee h) \quad G_f(w) = G_g(G_h(w)),$$

$$f(k) = \frac{1}{k!}\left(\frac{d}{dw}\right)^k G_f(w)\bigg|_{w=0}, \ k \in \mathbb{N}, \quad f(0) = g(0).$$

$$\mu_{f[k]} =: E((X)_k) = \left(\frac{d}{dw}\right)^k G_f(w)\bigg|_{w=1}, \ k \in \mathbb{N}, \quad \mu_{f[0]=1},$$

$G_f(w+1)$: "exponential generating function", factorial moment g.f.
$G_f(w)$: "ordinary generating function", probability g.f.

Weights in generating function.

Generating function $G(w; (\phi_k), (\omega_k))$ of $(\phi_k), k \in \mathbb{N}_0$, with "weight" $(\omega_k), k \in \mathbb{N}_0$:

$$G(w; \phi, \omega) := \sum_{0 \le k} \phi_k \omega_k w^k,$$

"ordinary type, "$\omega_k = 1$, "exponential type, "$\omega_k = 1/k!$.

$$\text{0-truncation of } X \sim \rho_k : \quad X^* := X|(X > 0)$$

$$X^* \sim \rho_k^* = \begin{cases} 0 & \text{if } k = 0, \\ \rho_k/(1-\rho_0) & \text{if } k > 0, \end{cases} \quad G(w; (\rho_k^*), \omega) = \frac{1}{1-\rho_0}(G(w; (\rho_k), \omega) - \rho_0).$$

$$\text{0-inflation of } X \sim \rho_k : \quad X^* = \begin{cases} 0 & \text{with prob.} \quad q = 1-p, \\ X & \text{with prob.} \quad p. \end{cases}$$

$$X^* \sim p\rho_k^* = \begin{cases} q + p\rho_0, & \text{if } k = 0, \\ p\rho_k, & \text{if } k > 0, \end{cases} \quad G(w; (\rho_k^*), \omega) = pG(w; (\rho_k), \omega) + q.$$

Ordinary and exponential generating function.

$$(\phi_k) := (\phi_k)_{k=0}^{\infty}, \quad (\psi_k) := (\psi_k)_{k=0}^{\infty},$$

ordinary generating fuction, formal power series.

$$G_o(w;(\phi_k)) := \sum_{k=0}^{\infty} \phi_k w^k = G_e(w;(k!\phi_k)),$$

$$\phi_k = [w^k]\{G_o(w;(\phi_k))\}$$

$$= \frac{1}{k!}\left(\frac{d}{dw}\right)^k G_o(w;(\phi_k)\Big|_{w=0}$$

exponential generating fuction), formal Taylor series

$$G_e(w;(\phi_k)) := \sum_{k=0}^{\infty} \frac{1}{k!}\phi_k w^k = G_o(w;(\phi_k/k!)),$$

$$\phi_k = k![w^k]\{G_e(w;(\phi_k))\}$$

$$= \left(\frac{d}{dw}\right)^k G_e(w;(\phi_k)\Big|_{w=0}$$

convolution

$$G_o(w;(\phi_k))G_o(w;(\psi_k)) = G_o(w;(\phi * \psi)_k),$$

$$(\phi * \psi)_k = \sum_{l=0}^{k} \phi_l \psi_{k-l},$$

$$\phi^{n*} = \phi^{(n-1)*} * \phi,\ n \in \mathbb{N}, \phi^{1*} := \phi.$$

multinomial convolution

$$G_e(w;(\phi_k))G_e(w;(\psi_k)) = G_e(w;(\phi \circ \psi)_k),$$

$$(\phi \circ \psi)_k = \sum_{l=0}^{k} \binom{l}{k} \phi_l \psi_{k-l},$$

$$\phi^{n\circ} = \phi^{(n-1)\circ} \circ \phi,\ n \in \mathbb{N}, \phi^{1\circ} := \phi.$$

2 Faà di Bruno formula and Bell polynomials

Faà di Bruno formula
for the differentiation of a composite function.

$$g, h \in \mathrm{C}^{\infty}, \quad f(x) = (g \circ h)(x) = g(h(x)),$$
$$h_0 = h(a),\ g_0 = g(b) = g(h(a)) = f_0,$$
$$g_k = \frac{d^k}{dy^k} g(y)|_{y=b}, \quad h_k = \frac{d^k}{dx^k} h(x)|_{x=a}, \quad f_k = \frac{d^k}{dx^k} f(x)|_{x=a},$$

$$f_n = \sum_{m=1}^{n} g_m \mathrm{B}_{n,m}(h_1, \ldots, h_{n-m+1}),\ n > 0, \quad f_0 = f(a) = g(h(a)).$$

$$\mathbf{B}_n(g;h) := \sum_{m=1}^{n} g_m\ \mathrm{B}_{n,m}(h), \quad g = (g_m),\ h = (h_m),\ n \in \mathcal{N}.$$

$\mathrm{B}_{n,m}(h)$: “Bell polynomials”

Faà di Bruno formula: D as a ball.

$$\begin{aligned}
f(t) =& g(h(t)), \quad g^{(k)} = g^{(k)}(h(t))\\
f^{(1)} =& g^{(1)}h^{(1)}\\
f^{(2)} =& g^{(2)}(h^{(1)})^2 + g^{(1)}h^{(2)}\\
f^{(3)} =& g^{(3)}(h^{(1)})^3 + 3g^{(2)}(h^{(1)})^2h^{(2)} + g^{(1)}h^{(3)}\\
f^{(4)} =& g^{(4)}(h^{(1)})^4 + 6g^{(3)}(h^{(1)})h^{(2)} +\\
& + g^{(2)}(4h^{(1)}h^{(3)} + 3(h^{(2)})^2) + g^{(1)}h^{(4)}
\end{aligned}$$

$g^{(2)}(\cdot)\ 4h^{(1)}h^{(3)} + 3(h^{(2)})^2$: 4=1+3=2+2

$$\begin{aligned} &(D_1\,h)(D_2D_3D_4\,h) + (D_2\,h)(D_1D_3D_4\,h) + (D_3\,h)(D_1D_2D_4\,h) \\ &+(D_4\,h)(D_1D_2D_3\,h) \\ &+(D_1D_2\,h)(D_3D_4\,h) + (D_1D_3\,h)(D_2D_4\,h) + (D_1D_4\,h)(D_2D_3\,h). \end{aligned}$$

Faà di Bruno formula: set partition form.

$$f^{(n)} = \sum_{m=1}^{n} g^{(m)} \sum^{*} (h^{(1)})^{s_1}(h^{(2)})^{s_2}\cdots(h^{(m)})^{s_m}, \quad \sum_{m=1}^{n} js_j = n, \ \sum_{m=1}^{n} s_j = m.$$

derivatives h^j: distinguishable, operator D's: indistinguishable,

W. C. Yang (2000) "Derivatives are essentially integer partitions", *Discrete Mathematics.*

Extensions: multi-composition, multi-variable and both.

(exponential partial) Bell polynomials. **Definition**

A sequence of 'variables' $(\phi_k)_{k=0}^{\infty}, \phi_0 = 0$, the exponential g.f. $\phi(t)$ (partial) Bell polynomials, $\mathrm{B}_{n,m}(\phi)$, $\phi = \phi(t)$ or $(\phi_m)_{m=0}^{\infty}$:

$$\Phi(t,u) := \exp(u\phi(t)) = 1 + \sum_{n\geq m\geq 1} \frac{u^m t^n}{n!}\,\mathrm{B}_{n,m}(\phi).$$

$$\frac{1}{n!}\mathrm{B}_{n,m}(\phi) = [t^n u^m]\{\exp(u\phi(t))\} = [t^n]\left\{\frac{1}{m!}\left(\sum_{k\geq 1}\phi_k\,\frac{t^k}{k!}\right)^m\right\}.$$

"total Bell polynomials", $\mathrm{Y}_n(\phi)$,

$$\Phi(t,1) = \exp(\phi(t)) = 1 + \sum_{n\geq 1}\mathrm{Y}_n(\phi_1,\ldots,\phi_n)\,\frac{t^n}{n!},$$

$$\mathrm{Y}_n(\phi) = \sum_{m=1}^{n}\mathrm{B}_{n,m}(\phi), \quad \mathrm{Y}_0(\phi) := 1,$$

Bell polynomials, more explicitly.

$$\mathcal{P}_{n,m} := \{s = (s_1,\ldots,s_n) \in \mathbb{N}_0^n : \sum_{n\geq j\geq 1} js_j = n; \sum_{n\geq j\geq 1} s_j = m\}$$

$$\mathcal{P}_n := \bigcup_{n\geq m\geq 1}\mathcal{P}_{n,m}.$$

$$\mathrm{B}_{n,m}(\phi) = \mathrm{B}_{n,m}(\phi_1, \ldots, \phi_{n-m+1}) = \sum_{s \in \mathcal{P}_{n,m}} \pi(n,s) \prod_{j=1}^{n} \phi_j^{s_j}.$$

$$s = (s_1, \ldots, s_n) \in \mathcal{P}_n, \quad \pi(n,s) = \frac{n!}{\prod_{j=1}^{n} s_j! \, (j!)^{s_j}}$$

Bell polynomials, some properties.

1. Bell polynomials satisfy the recurrence equation,

$$\mathrm{B}_{n,m}(\phi) = \frac{1}{m} \sum_{j=m-1}^{n-1} \binom{n}{j} \phi_{n-j} B_{j,m-1}(\phi), \quad 0 \le m \le n,$$

$$B_{0,0}(\phi) = 1,\ B_{n,0}(\phi) = B_{0,m}(\phi) = 0,\ \text{if } (n,m) \ne (0,0),$$

$$B_{n,1}(\phi) = \phi_n,\ B_{n,n}(\phi) = \phi_1^n.$$

2. $\mathrm{B}_{n,m}$ is a polynomial of order m, of $\phi_1, \ldots, \phi_{n-m+1}$ with positive integer coefficients.

3. $\mathrm{B}_{n,m}((1^k)) = \sum_{s \in \mathcal{P}_{n,m}} \pi(n,s) = \left\{ \begin{matrix} n \\ m \end{matrix} \right\}$

4. $$\mathrm{B}_{n,m}(ab\phi_1, ab^2\phi_2, \ldots, ab^n\phi_n) = a^m b^n \mathrm{B}_{n,m}(\phi_1, \phi_2, \ldots, \phi_n).$$

   with g.f. $\exp(au\phi(bt))$.

5. Combinatorial meaning of $\pi(n,s)$: Put n indistinguishable balls into m distinguishable urns without empty urn, so that the number of urns with j balls is $s_j, 1 \le j \le n$, ($\sum_j s_j = m, \sum_j js_j = n$). The number of possible arrangements of balls is $\pi(n,s), s \in \mathcal{P}_{n,m}$

3 General compound distributions

Compound distributions in general.

$$G_f(w) = G_g(G_h(w)),, \quad f_0 = g_0,\ h_0 = 0$$

$$f_n = \frac{1}{n!} \mathbf{B}_n((k!g_k), (k!h_k)) = \frac{1}{n!} \sum_{1 \le m \le r} m! g_m \mathrm{B}_{n,m}((k!h_k)),\ k \in \mathbb{N},$$

$$\mu_{f[r]} = \mathbf{B}_n((\mu_{g[k]}), (\mu_{h[k]})) = \sum_{1 \le m \le r} \mu_{g[m]} \mathrm{B}_{n,m}((\mu_{h[k]})).$$

For compound Poisson, since $k!g_k = e^{-\lambda}$ is independent of k

$$f_n = \frac{e^{-\lambda}}{n!} \sum_{1 \le m \le n} \mathbf{B}_{n,m}((k!h_k)),$$

$$\mu_{f[r]} = \frac{e^{-\lambda}}{n!} \sum_{1 \le m \le r} \mathrm{B}_{n,m}((\mu_{h[k]})).$$

Incidentally,

$$P\{Z_1 + \cdots + Z_m = n\} = \frac{m!}{n!} \mathrm{B}_{n,m}((k!h_k)), \text{ if } h_0 = 0.$$

Random partition of integer.

$$X = Y \bigvee Z, \quad (\text{or} \quad f = g \bigvee h) \quad G_f(w) = G_g(G_h(w)),$$

In compoud Poisson distributions, if Y and/or $Z = (Z_1, \ldots, Z_y)$ are observable,

$$P\{Y = y | X = x\} = \frac{\lambda^y \mathrm{B}_{x,y}(\phi)}{\mathrm{Y}_x(\lambda\phi)} = \frac{\mathrm{B}_{x,y}(\lambda\phi)}{\mathrm{Y}_x(\lambda\phi)}, \quad \phi_k = k!h_k,$$

$$P\{S = s | X = x, Y = y\} = \frac{1}{\mathrm{B}_{x,y}(\phi)} \pi(x, s) \prod_{j=1}^{n} \phi_j^{s_j},$$

$$s \in \mathcal{P}_{x,y},\ 1 \le y \le x,$$

$$S = (S_1, \ldots, S_X), S_j := \sum_{Y \ge i \ge 1} I[Z_i = j],\ X \ge j \ge 1.$$

$P\{S = s | X = x\}$ is a random partition of x, and ¶{Y=y—X=x} is the number of cluster (species) given total number of individuals. In general compound distributions,

$$P\{Y = y | X = x\} = \frac{g_y \mathrm{B}_{x,y}((h_k))}{\mathbf{B}_x((k!g_k), (k!h_k))},$$

$$P\{S = s | X = x, Y = y\} = \frac{1}{\mathrm{B}_{x,y}((h_k))} \pi(x, s) \prod_{j=1}^{n} h_j^{s_j}, \quad s \in \mathcal{P}_{x,y}.$$

4 Typical compounding distributions

Compound logarithmic series and compound binomial.

1. Compound logarithmic series distributions

$$G_g(w) = \log(1-\xi w)/\log(1-\xi),\ \xi \in (0,1),$$

$$f_n = \frac{1}{-\log(1-\xi)}\mathbf{B}_n(((k-1)!\xi^k),\ \phi)$$

$$= \frac{1}{-\log(1-\xi)}\sum_{k=1}^{n}(k-1)!\xi^k \mathrm{B}_{n,k}(\phi),\ \phi_k = k!h_k.$$

"logaritmic Bell polynomials" for a specific value $(\xi = -1)$:

$$[u^k t^n]\{\log(1+u\phi(t))\} = \sum_{k=1}^{n}(-1)^{k-1}(k-1)!\mathrm{B}_{n,k}(\phi)$$

$$= \mathbf{B}_n(((-1)^{k-1}(k-1)!),\ \phi) =: \mathrm{L}_n(\phi).$$

2. Compound binomial distributions

$$G_g(w) = (1-\xi+\xi w)^m,\ \xi \in (0,1),$$

$$f_n = (1-\xi)^m \mathbf{B}_n\left(\left((m)_k\left(\frac{\xi}{1-\xi}\right)^k\right),\ \phi\right)$$

$$= (1-\xi)^m \sum_{k=1}^{\min(n,m)}(m)_k\left(\frac{\xi}{1-\xi}\right)^k \mathrm{B}_{n,k}(\phi),$$

$$\phi_k = k!h_k, \quad (m)_k = \prod_{0\le j<k}(m-j).$$

"potential Bell polynomials": 2^m times of specific value $(\xi = 1/2)$:

$$[u^k t^n]\{(1+\phi(t))^m\} = \sum_{k=1}^{\min(n,m)}(m)_k \mathrm{B}_{n,k}(\phi) = \mathbf{B}_n(((m)_k),\ \phi).$$

An application of compound binomial: Convolution.

Convolution of nonnegative integer value disributions.
$Z = (Z_1, \ldots, Z_m)$ i.i.d.

$$P\{Z_i = k\} = \rho_k,\ k \in \mathcal{N}_0, \quad G_Z(w) = \sum_{z=0}^{\infty}\rho_z w^z,$$

$$X = \mathsf{Bn}\ (m, 1-\rho_0) \bigvee Z|(Z>0) \quad X \sim Z_1 + \cdots + Z_m.$$

$$P\{X=n\} = [w^n]\{(G_Z(w))^m\}$$

$$= \begin{cases} \rho_0^m, & n = 0, \\ \frac{1}{n!}\sum_{k=1}^{\min(n,m)}(m)_k \rho_0^{m-k}\mathrm{B}_{n,k}(1!\rho_1, 2!\rho_2, \ldots), & n > 0. \end{cases}$$

5 Compound Poisson distributions

Compound Poisson: compounding truncated distributions.

$$\text{nonnegative integer valued } \mathsf{H} \;:(h_k)_{k=0^\infty},$$
$$\text{0-truncated distributions } \mathsf{TH} \;:h_k^* = h_k/(1-h_0),\; k>0,\; h_0^*=0,$$
$$\mathsf{Po}\,(\lambda) \bigvee \mathsf{H} \;= \mathsf{Po}\,((1-h_0)\lambda) \bigvee \mathsf{TH} \quad \Longleftrightarrow \quad \mathsf{Po}\,(\lambda) \bigvee \mathsf{TH} \;= \mathsf{Po}\,(\lambda/(1-h_0)) \bigvee \mathsf{H}\,.$$

Compound Poisson: negative binomial distributions.

$$\mathsf{NgBn}\,(\eta,p) = \mathsf{Po}\,(\lambda) \bigvee \mathsf{LgSer}\,(p), \quad \eta>0,\; 0<p<1.$$
$$\eta = \frac{\lambda}{-\log(1-p)},\; \lambda = -\eta\log(1-p),\; e^{-\lambda} = (1-p)^\eta.$$
$$\mathsf{Po}\,(\lambda):\; \exp(\lambda(v-1)),$$
$$\mathsf{LgSer}\,(p):\; v = \log(1-pw)/\log(1-p),$$
$$\mathsf{NgBn}\,(\eta,p):\; \exp\left(-\eta\log\frac{1-pw}{1-p}\right) = \left(\frac{1-pw}{1-p}\right)^{-\eta}.$$
$$\phi_k := k!h_k = \frac{(k-1)!}{-\log(1-p)}p^k = (k-1)!p^k\frac{\eta}{\lambda},$$
$$\lambda^m \mathrm{B}_{n,m}((\phi_k)) = \eta^m p^n \mathrm{B}_{n,m}(((k-1)!)) = \eta^m p^n \begin{bmatrix} n \\ m \end{bmatrix}.$$
$$f_n = \frac{e^{-\lambda}}{n!}\sum_{1\le m\le n} \lambda^m \mathrm{B}_{n,m}((\phi_k)) = \frac{(\eta|-1)_n}{n!}(1-p)^\eta p^n.$$

$Z_i \sim \mathsf{LgSer}\,(p)$ i.i.d. $T_m = Z_1 + \cdots + Z_m$,

$$P\{T_m = n\} = \frac{m!}{(-\log(1-p))^m}\begin{bmatrix} n \\ m \end{bmatrix} p^n,\; m\le n,$$

$\begin{bmatrix} n \\ m \end{bmatrix}$ unsigned Stirling of the first kind:

$$(x|-1)_n = \sum_{1\le m\le n} \begin{bmatrix} n \\ m \end{bmatrix} x^m, \quad 1\le m\le n,$$

Compound Poisson: Neyman Type A.

$\mathsf{Po}\,(\lambda_0 = \lambda/(1-\mathrm{e}^{-\nu})) \bigvee \mathsf{Po}\,(\nu)$, $\quad \mathsf{Po}\,(\lambda) \bigvee \mathsf{TPo}\,(\nu)$, $\quad \mathsf{Po}\,(\nu j) \bigwedge_j \mathsf{Po}\,(\lambda_0)$,

$\mathsf{TPo}\,(\nu):\ \dfrac{e^{\nu w}-1}{e^{\nu}-1}$, $\quad \mathsf{NeymanA}\,(\lambda_0,\nu):\ \exp(\lambda_0(\exp(\nu(w-1))-1))$.

$$\phi_k := k!h_k = \frac{\nu^k}{e^{\nu}-1},$$

$$\lambda^m \mathrm{B}_{n,m}((\phi_k)) = \frac{\lambda_0^m(1-e^{-\nu})^m}{(e^{\nu}-1)^m}\nu^n \mathrm{B}_{n,m}((1)) = (\lambda_0 e^{\nu})^m \nu^n \mathrm{B}_{n,m}((1))$$

$$f_n = \frac{e^{-\lambda_0}\nu^n}{n!}\sum_{j=0}^{\infty}\frac{(\lambda_0 e^{-\nu})^j j^n}{j!} = \frac{e^{-\lambda}\nu^n}{n!}\sum_{k=1}^{n}\left\{ {n \atop k} \right\}(\lambda_0 e^{-\nu})^k,\ n\in\mathcal{N}_0,\quad f_0 = e^{-\lambda},$$

$$E(X^{\underline{r}}) = \nu^r \sum_{k=1}^{r}\left\{ {r \atop k} \right\}\lambda^k,\ r\in\mathcal{N}_0.$$

Compound Poisson: Thomas.

$X \sim \mathsf{Po}\,(\lambda) \bigvee \mathsf{ShiftedPo}\,(\nu)$, $\quad Z \sim \mathsf{ShiftedPo}\,(\nu) \Leftrightarrow Z = 1 + Z^*,\ Z^* \sim \mathsf{Po}\,(\nu)$.

$\mathsf{Po}\,(\lambda,\nu):\ \exp(\lambda(w\exp(\nu(w-1))-1)) = \exp(\lambda w \exp(\nu w - (1+\nu)))$,

$$\phi_k := k!h_k = k\nu^{k-1}e^{-\nu},\quad \lambda^m \mathrm{B}_{n,m}((\phi_k)) = \lambda^m \nu^{n-k} e^{-k\nu}\mathrm{B}_{n,m}((k)),$$

$$f_n = \frac{e^{-\lambda}\nu^n}{n!}\sum_{k=1}^{n}\binom{n}{k}(\nu k)^{n-k}(\lambda e^{-\nu})^k.$$

$$\mathrm{B}_{n,k}((k)) = \binom{n}{k}k^{n-k},\quad 0\le k\le n,\quad \text{idempotent numbers}$$

Compound Poisson: cumulants.

negative binomial$\mathsf{NgBn}\,(a,h), h = p/(1-p),\ p$: failure probability,

$$\kappa_r = a\sum_{k=1}^{r}\left\{ {r \atop k} \right\}(k-1)!h^k.$$

Neyman type A$\mathsf{NeyA}\,(\lambda,\nu)$,

$$\kappa_r = \lambda\sum_{k=1}^{r}\left\{ {r \atop k} \right\}\nu^k.$$

Thomas$\mathsf{Thomas}\,(\lambda,\nu)$,

$$\kappa_r = \lambda\sum_{k=1}^{r}\left\{ {r \atop k} \right\}(\nu+k)^{k-1}.$$

Some Bell polynomials.

$$\mathrm{B}_{n,k}((k-1)!) = \begin{bmatrix} n \\ k \end{bmatrix}, \quad \text{unsigned Stirling numberof the first kind,}$$

$$\mathrm{B}_{n,k}((1^k)) = \begin{Bmatrix} n \\ k \end{Bmatrix}, \quad \text{Stirling numberof the second kind,}$$

$$\mathrm{B}_{n,k}((k)) = \binom{n}{k} k^{n-k}, \quad \text{idempotent number,}$$

$$\mathrm{B}_{n,k}((k!)) = \binom{n-1}{k-1} \frac{n!}{k!}, \quad \text{Lah number,}$$

$$\mathrm{B}_{n,k}((n)_{k-1}) = \sum_{0\le k\le m\le n} \begin{bmatrix} n \\ m \end{bmatrix} \begin{Bmatrix} m \\ k \end{Bmatrix} (-1)^{n-m} n^m,$$

$$\mathrm{B}_{n,k}((r|-1)_{k-1}) = \sum_{0\le k\le m\le n} \begin{bmatrix} n \\ m \end{bmatrix} \begin{Bmatrix} m \\ k \end{Bmatrix} r^m.$$

6 Typical compounded distributions

Compounded Engen distributions.

negative binomial $\mathsf{NgBn}\,(p,a), 0<p<1, 0<a, \quad p:$ failure probability.
zero-truncated $\mathsf{TNgBn}\,(p,a), 0<p<1, 0<a,$
extended zero-truncated $\mathsf{TNgBn}\,(p,a), -1\le p<0, 0<a$

$$\frac{1}{1-(1-p)^a}\left(\left(\frac{1-p}{1-pw}\right)^a - (1-p)^a\right) = \frac{(1-pw)^{-a}-1}{(1-p)^{-a}-1}.$$

7 Moments and cumulants

Moments and cumulants. ex-log relation.

moments around the origin(raw moments) $(\alpha_r)_{r=1}^{\infty}$
cumulans $(\kappa_r)_{r=1}^{\infty}$ m.g.f. and c.g.f.

$$\alpha(t) = 1 + \sum_{r=1}^{\infty} \alpha_r t^r/r!, \quad \kappa(t) = \sum_{r=1}^{\infty} \kappa_r t^r/r!, \quad \kappa(t) = \ln(\alpha(t))$$

$\alpha(t) = \exp(\kappa(t))$ is a total Bell polynomial,

$$\alpha_n = \sum_{k=1}^{n} \mathrm{B}_{n,k}(\kappa) = \mathrm{Y}_n(\kappa), \quad n\in\mathbb{N}.$$

Conversely, since $\kappa(t) = \log \alpha(t) = \log(1+(\alpha(t)-1))$ is a logarithmic type Bell polynomial,

$$\kappa_n = \sum_{k=1}^{n} (-1)^{k-1}(k-1)!\mathrm{B}_{n,k}(\alpha) = \mathrm{L}_n(\alpha), \quad n \in \mathbb{N}.$$

This relationship holds between factorial moments and factorial cumulants (f.m.g.f. and f.c.g.f.)

Cumulants of compoud distributions.

$$X = Z_1 + \cdots + Z_Y \quad G_f(w) = G_g(G_h(w)),$$

Among cumulants $\kappa_{f,r}, \kappa_{g,r}$, and $\kappa_{h,r}$.

$$\kappa_{f,r} = \sum_{k=1}^{r} \kappa_{g,k} \mathrm{B}_{n,k}(\kappa_h), \quad \kappa_h = (\kappa_{h,1}, \kappa_{h,2}, \dots). \tag{1}$$

This relationship holds among factorial cumulants $\kappa_{f[r]}, \kappa_{g[r]}$, and $\kappa_{h[r]}$.

8 Generalized Stirling numbers

Generalized Stirling numbers. **Definition**.

(Hsu and Shiue, 1998) Generalized Stirling numbers $S_{n,m}$ for variables α, β, r:

$$(t-r|\alpha)_n \equiv \sum_{k=0}^{n} S_{n,k}(\alpha,\beta,r)\ (t|\beta)_k, \quad (t|\beta)_k = \prod_{k\ge j\ge 1} (t-(j-1)\beta).$$

$S_{n,m}(\alpha,\beta,r)$ the first kind $\quad S_{n,k}(\beta,\alpha,-r)$ the second kind

$$S_{n+1,k}(\alpha,\beta,r) = (k\beta - n\alpha - r)\ S_{n,k}(\alpha,\beta,r) + S_{n,k-1}(\alpha,\beta,r),$$
$$0 \le k \le n+1; \quad S_{0,0}(\alpha,\beta,r) = 1;$$
$$S_{n,k}(\alpha,\beta,r) = 0, \quad \text{unless} \quad k \in \{0,1,\dots,n\}.$$

Generalized Stirling numbers. An explicit expression.

$$S_{n,k}(\alpha,\beta,r) \quad = \quad \sum_{0\le k\le l\le m\le n} \begin{bmatrix} n \\ m \end{bmatrix} \binom{m}{l} \left\{ \begin{matrix} l \\ k \end{matrix} \right\} (-\alpha)^{n-m}(-r)^{m-l}\beta^{l-k}.$$

$$= \left(\begin{bmatrix} n \\ m \end{bmatrix} (-\alpha)^{n-m} \right) \cdot \left(\binom{m}{l} (-r)^{m-l} \right) \cdot \left(\left\{ \begin{matrix} l \\ k \end{matrix} \right\} \beta^{l-k} \right).$$

Generalized Stirling numbers and Bell polynomials.

Proposition Generalized Stirling numbers form a specific subgroup of Bell polynomials.

Table 1: p.g.f. generating $S_{n,m}(\alpha, \beta, 0) = S_{n,m}(\beta, \alpha, 0)$ by Compound Poisson

$\phi(t)$	$\phi_k, k \in \mathbb{N}$	(α, β)	p.g.f. $\phi(\theta t)/\phi(\theta)$
$e^t - 1,$	$1,$	$(1, 0),$	TPo (θ), $0 < \theta$, $\theta = 1.$
$-\log(1-t),$	$(k-1)!,$	$(-1, 0),$	LgSer (θ), $0 < \theta < 1$, $\theta = 1.$
$(1+t)^m - 1,$	$(m)_k, m \in \mathbb{N},$	$\beta/\alpha = m,$	TBn $(m, \theta/(1-\theta)$, $0 < \theta, \theta = 1/2.$
$(1-t)^{-r} - 1$	$(r\vert -1)_k, 0 < r,$	$\beta/\alpha = -r$, $\alpha < 0 < \beta$	TNgBn (r, θ), $0 < \theta < 1,$
$1 - (1-t)^r,$	$-(-r\vert -1)_k, 0 < r < 1$	$\beta/\alpha = r$, $0 < \beta < \alpha,$	ETNgBn (r, θ), $0 < \theta < 1; \theta = 1$, Sby (r)

9 Lagrangian probability distributions

Lagrangian probability distributions.

Compound Poisson distributions: “stopped sum”,
instance of Y(t), Poisson process, (pure birth processes)
Lagrangian probability distributions: size of nodes
at the distinction time of a branching process (birth and death processes)
Number of nodes at the n-th generation: Y_n, $G_n(w)$
Number of branches at each node: Z with p.g.f. $G_Z(w)$, $G_Z(0) > 0, i.i.d.$
? p.g.f. $R(w)$ of $X = \sum_{n=0}^{\infty} Y_n$,
$E(Z) < 1 \iff X$: proper distribution and $E(X) < \infty$.
If $P\{Y_0 = 1\} = 1$,

$$R(w) = wG_Z(R(w)) \quad \Leftrightarrow \quad R(w) = (w/G_Z(w))^{\leftarrow}$$
$$X - 1 \sim \mathcal{L}(G_Z)(w) := w^{-1}R(w) = w^{-1}(w/G_Z(w))^{\leftarrow}.$$

Further if Y_0 is random with p.g.f. $G_Y(w)$, p.g.f. of X:

$$\mathcal{L}(G_Z; G_Y)(w) = G_Y(R(w)) = G_Y((w/G_Z(w))^{\leftarrow}).$$

Bell polynomial of the inverse of p.g.f.

Let a p.g.f. and its inverse be denoted by

$$\breve{\phi}(w) := \sum_{k=1}^{\infty} \phi_k w^k, \quad \phi_0 = 0,\ \phi_1 \neq 0,$$
$$(\breve{\phi}^{-1} \circ \breve{\phi})(t) = (\breve{\phi} \circ \breve{\phi}^{-1})(t) = w.$$

Then,

$$[t^n]\{G_o(t;\breve{\phi}^{-1})^k\} = \frac{k}{n}[t^{n-k}]\left\{\left(\frac{\breve{\phi}(t)}{t}\right)^{-n}\right\}.$$

If $k=1$, using $\delta_{11}=1$, $\delta_{k1}=0, k>1$,

$$[t^n]\{G_o(t;\breve{\phi}^{-1})\} = \sum_{k=1}^{n-1}(-1)^k\phi_1^{-n-k}\mathrm{B}_{n+k-1,k}(((1-\delta_{k1})\phi_k)).$$

Future works.

- comparison between compound distributions
- asymptotic theory
- "discrete stable distributions"
- semi parametric models
- software development
- other stocastic mechanisms
- extensions to multivariate distributions, and to multiple composition

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Table 2: Kerov's argument, $X = Y \bigvee Z$.

Z	Y	p_2 specification	X	Kolchin model
ETNgBn$(p_1, -r_1)$,	NgBn$(p_2, r_2), r_2 > 0$	$p_2 = 1 - (1-p_1)^{-r_1}$	NgBn$(p_1, r_1 r_2)$	$0 < \alpha < 1, \theta > 0$
$r_1 > 0$,	TNgBn$(p_2, r_2), r_2 > 0$,	- do -	TNgBn$(p_1, r_1 r_2)$	$0 < \alpha < 1, \theta > 0$
	ETNgBn$(p_2, -r_2)$,	- do -	ETNgBn$(p_1,$	$0 < \alpha < 1,$
	$r_2 > 0$,		$-r_1 r_2)$	$-\alpha < \theta < 0$
	LgSer(p_2)	$p_2 = p_1, \forall r_1 > 0$,	LgSer(p_1)	$0 < \alpha < 1, \theta = 0$
Sby$(r_1), r_1 > 0$	Sby$(r_2), r_2 > 0$,	no specification	Sby$(r_1 r_2)$	$\alpha = 1, -\alpha < \theta < 0$
LgSer(p)	Po(λ)	$\lambda = -r \log(1-p)$	NgBn(p, r)	$\alpha = 0, \theta > 0$
TPo(λ)	Bn(m_2, p_2)	$p_2 = 1 - e^{-\lambda}$	Po(λm_2)	occupancy
TNgBn$(p_1, -r_1)$	Bn(m_2, p_2)	$p_2 = 1 - (1-p_1)^{-r_1}$	NgBn(m_2, p_1)	$\theta/\alpha = -1, -2, \ldots$
TBn(p_1, m_1)	Bn(m_2, p_2)	$p_2 = 1 - (1-p_1)^{m_1}$	Bn$(m_1 m_2, p_1)$	MvHg

Table 3: Partitions, Bell monomials, Stirling numbers of the 2nd kind, and Bell numbers.

n	k	s_1	s_2	s_3	s_4	s_5	monomials		$B_{n,k}(\phi)$	$\left\{ {n \atop k} \right\}$	B_n
1	1	1					1	ϕ_1	ϕ_1	1	1
2	1	0	1				1	ϕ_2	ϕ_2	1	
	2	2	0				1	ϕ_1^2	ϕ_1^2	1	2
3	1	0	0	1			1	ϕ_3	ϕ_3	1	
	2	1	1	0			3	$\phi_1\phi_2$	$3\phi_1\phi_2$	3	
	3	3	0	0			1	ϕ_1^3	ϕ_1^3	1	5
4	1	0	0	0	1		1	ϕ_4	ϕ_4	1	
	2	1	0	1	0		4	$\phi_1\phi_3$			
	2	0	2	0	0		3	ϕ_2^2	$4\phi_1\phi_3 + 3\phi_2^2$	7	
	3	2	1	0	0		6	$\phi_1^2\phi_2$	$6\phi_1^2\phi_2$	6	
	4	4	0	0	0		1	ϕ_1^4	ϕ_1^4	1	15
5	1	0	0	0	0	1	1	ϕ_5	ϕ_5	1	
	2	1	0	0	1	0	5	$\phi_1\phi_4$			
	2	0	1	1	0	0	10	$\phi_2\phi_3$	$5\phi_1\phi_4 + 10\phi_2\phi_3$	15	
	3	2	0	1	0	0	10	$\phi_1^2\phi_3$			
	3	1	2	0	0	0	15	$\phi_1\phi_2^2$	$10\phi_1^2\phi_3 + 15\phi_1\phi_2^2$	25	
	4	3	1	0	0	0	10	$\phi_1^3\phi_2$	$10\phi_1^3\phi_2$	10	
	5	5	0	0	0	0	1	ϕ_1^5	ϕ_1^5	1	52