Smile Curve and Local Volatility

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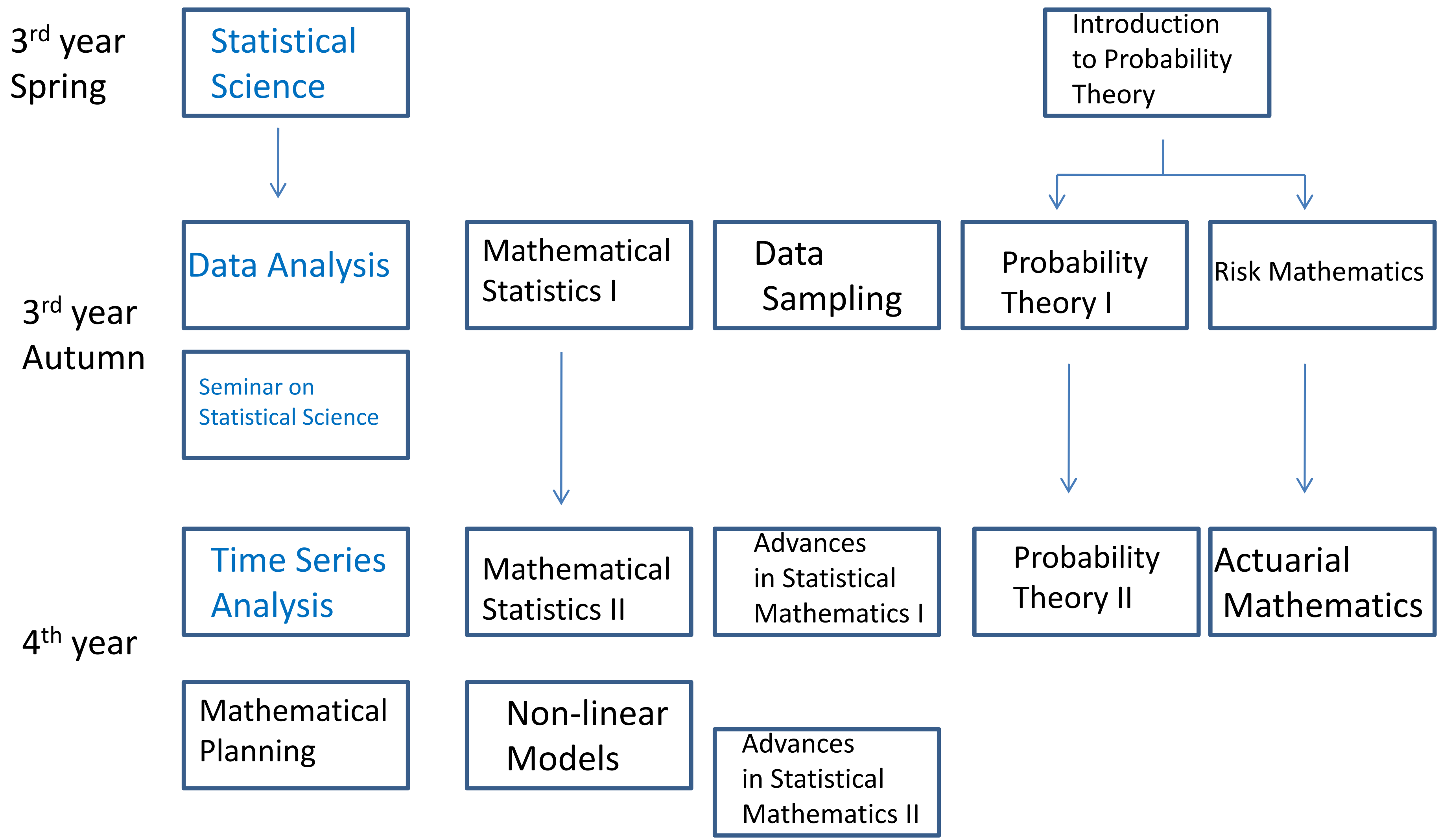


Graduate School

Data Literacy

by Ritei Shibata

Data Sciences

by Invited professors

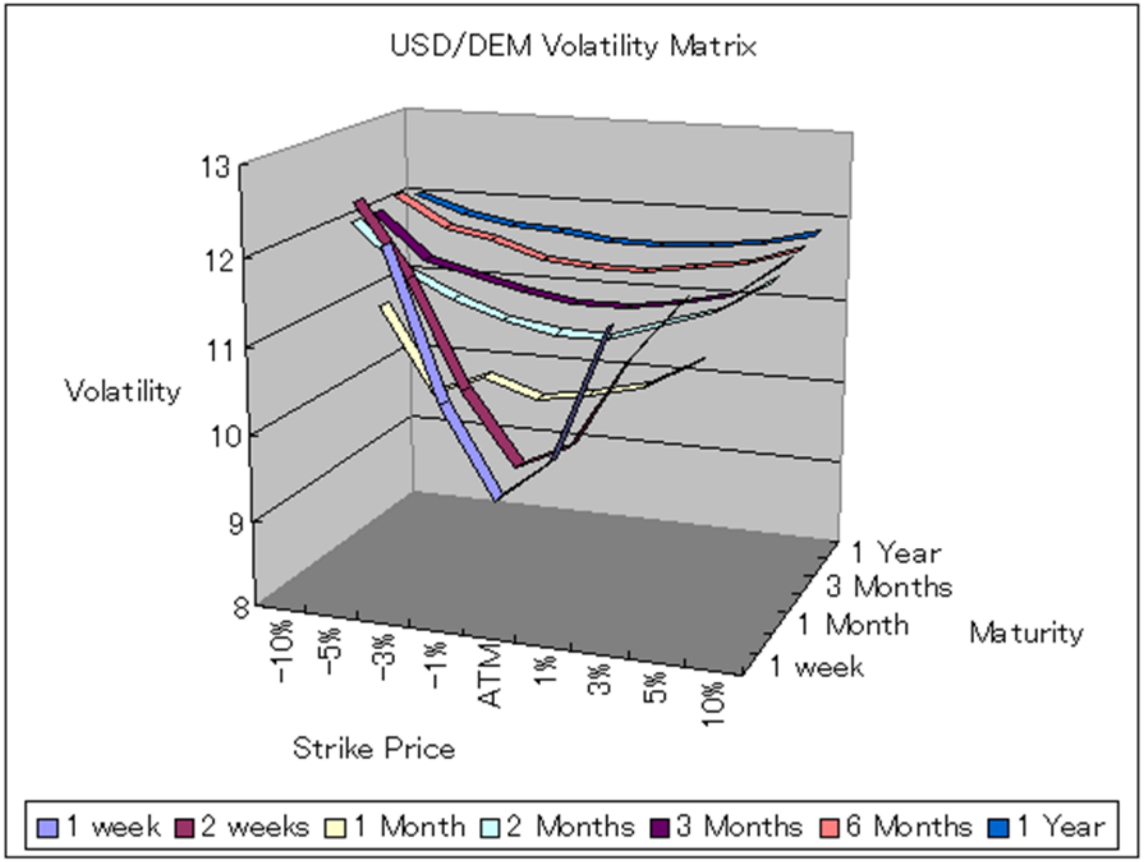
Mathematical Finance

by Ritei Shibata and Makoto Maejima

Advances in Mathematical Finance

by Nobuhiro Nakamura (Hitotsubashi University)

Financial Smiles



http://www.fpohkumi.com/FIN702.htm

A smile is a curve that sets everything straight

Icon	Meaning
:-)	classic smile with nose
:-(	classic frown with nose (Unicode: ☹)
:)	classic smile without nose
:(	classic frown without nose

Option Price

$$c(t,x) = e^{-rt} \int_{\log(x)}^{\infty} (e^y - x)\, f_t(y)\, dy : \ Call\, Option\, Price$$

$$t : \ Time\, of\ \ Execution\, (Maturity)$$

$$x : \ \ Execution\, Price$$

$$f_t : \ \ Distribution\, of\ \log(S_t / S_0)$$

$$S_t : Underlying\ Asset\ Price$$

Black-Scholes

Assumption

$$dS_t = r\,S_t\,dt + \sigma\,S_t\,dB_t \quad (\,log\,normal\;\; process)$$

$$c(t,x) = e^{-rt}\int_{\log(x)}^{\infty} (e^y - x)\,f_t(y)\,dy$$

$$\Longrightarrow \quad = S_0\,\Phi(-\lambda - \sigma\sqrt{t}) - x\,e^{-rt}\,\Phi(-\lambda),$$

$$\lambda = \frac{\log(x/S_0) - (r - \sigma^2/2)t}{\sigma\sqrt{t}}$$

Implied Volatility

$$c(t,x) = S_0\,\Phi(-\lambda - \sigma\sqrt{t}) - x\,e^{-rt}\,\Phi(-\lambda) : \textit{Black Scholes Formula}$$

$$\lambda = \frac{\log(x/S_0) - (r - \sigma^2/2)t}{\sigma\sqrt{t}}$$

$$c(t,x) : \textit{Option Price at } t = 0$$

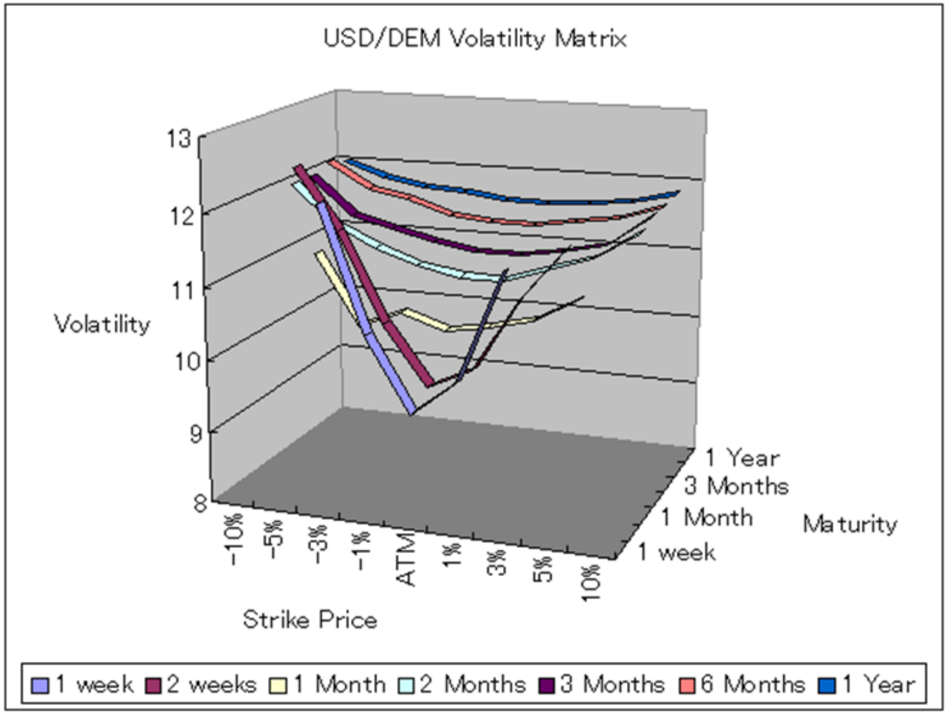
$$\Longrightarrow \quad \sigma_{ip}(t,x) : \textit{Implied Volatility}$$

Smile or not Smile

Assumption

$$dS_t = r\,S_t\,dt + \sigma\,S_t\,dB_t \quad (\ log\ normal\ process) \quad \longrightarrow \quad \sigma_{ip}(t,x) \equiv \sigma$$

Reality



What is wrong

Assumption

$$dS_t = r\, S_t\, dt + \sigma\, S_t\, dB_t \quad (\, log\; normal\; process)$$

$$\Longleftrightarrow \quad \log(S_t / S_0) \sim N((r - \sigma^2/2)t,\; \sigma^2 t\,)$$

$$\Longleftrightarrow \quad f_t \;:\; Normal\; density$$

The distribution?

$$c(t,x) = e^{-rt} \int_{\log(x)}^{\infty} (e^y - x)\, f_t(y)\, dy$$

$$\Longrightarrow \quad \frac{\partial}{\partial x} c(t,x) = -e^{-rt} \left(1 - F_t(\log x) \right) \; for\ x > 0$$

$$\Longrightarrow \quad \boxed{\frac{\partial^2}{\partial x^2} c(t,x) = -e^{-rt} \, \frac{1}{x} \, f_t(\log x) \;\; for\ x > 0}$$

Estimation of the distribution

$$\frac{\partial^2}{\partial x^2} c(t,x) = -e^{-rt} \frac{1}{x} f_t(\log x) \quad for\ x > 0$$

$$c(t,x) = S_0\,\Phi(-\lambda - \sigma_{ip}(t,x)\sqrt{t}) - x\,e^{-rt}\,\Phi(-\lambda),$$

$$\lambda = \frac{\log(x/S_0) - (r - \sigma_{ip}(t,x)^2/2)t}{\sigma_{ip}(t,x)\sqrt{t}}$$

$$\Downarrow$$

$$\sigma_{ip}(t,x) \Rightarrow f_t$$

Some Attempts

- Shimko(1993), RISK 6, 33-37
  - quadratic model for implied volatility
- Brunner and Hafner (2003), J. Compt. Finance
  - mixture of log normals for the distribution

$$\frac{\partial^2}{\partial x^2} c(t,x) = e^{-rt} \ \frac{1}{x} f_t(\log x) \ \ for\, x > 0$$

$$c(t,x) = S_0\, \Phi(-\lambda - \sigma_{ip}(t,x)\sqrt{t}) - x\, e^{-rt}\, \Phi(-\lambda)$$

⬆

Assumption

$$dS_t = r\, S_t\, dt + \sigma\, S_t\, dB_t \ \ (\ log\ normal\ \ process)$$

Local Volatility Model

$$dS_t = r\,S_t\,dt + \sigma(t, S_t)\,S_t\,dB_t$$

$$\tilde{c}(t,x) = e^{rt} c(t,x)$$

$$\Downarrow$$

$$\frac{\partial}{\partial t}\tilde{c}(t,x) = \frac{x}{2}\left(\sigma^2(t,x)\,f_t(\log x)\right)$$

$$\Downarrow \qquad \frac{\partial^2}{\partial x^2}\tilde{c}(t,x) = \frac{1}{x}\,f_t(\log x) \quad \text{for } x > 0$$

$$\sigma^2(t,x) = \frac{2\dfrac{\partial}{\partial t}\tilde{c}(t,x)}{x^2\dfrac{\partial^2}{\partial x^2}\tilde{c}(t,x)}\;;\quad Dupire(1994)$$

Estimation of local volatility

Assumption

$$dS_t = r\,S_t\,dt + \sigma(t, S_t)\,S_t\,dB_t$$

$$\tilde{c}(t,x) = e^{rt} c(t,x)$$

$$\frac{\partial}{\partial t}\tilde{c}(t,x) = \frac{x}{2}\left(\sigma^2(t,x)\,f_t(\log x)\right)$$

Nicolas Rousseanu (2007)

$$Scaling\ \ Assumption: \frac{\log(S_t/S_0)-\mu(t)}{s(t)} \overset{d}{=} \frac{\log(S_1/S_0)-\mu(1)}{s(1)},\ t>0$$

$$Gram-Charlier\ Expansion\ of\ \ f_t$$

$$\frac{\partial}{\partial t}\tilde{c}(t,x),\ \ \mu(t),\ s(t),\ \ Cummulants\ of\ \ \frac{\log(S_t/S_0)-\mu(t)}{s(t)} \Rightarrow \sigma^2(t,x)$$

$$\sigma_{ip}(t,x) : Smile\ Surface$$

$$\Longleftrightarrow \quad \frac{\partial}{\partial t}\tilde{c}(t,x)$$

$$\Longleftrightarrow \quad \sigma^2(t,x) : Local\ Volatility$$

$$\frac{\partial}{\partial t}\tilde{c}(t,x) = \frac{x}{2}\left(\sigma^2(t,x)\, f_t(\log x)\right) > 0\ ?$$

$$\frac{\partial^2}{\partial x^2}\tilde{c}(t,x) = \frac{1}{x}\, f_t(\log x) > 0\ ?$$

Our results

$$(A)\ \frac{\partial}{\partial t}\log\sigma_{ip}(t,x) > -\frac{1}{2t} \quad \Leftrightarrow \quad \frac{\partial}{\partial t}\tilde{c}(t,x) > 0$$

$$(B)\ 1 + x\left(\frac{\partial}{\partial x}\log\sigma_{ip}(t,x)\right)\left(\frac{t}{2}\sigma_{ip}^2(t,x) - \log(x/S_0)\right) > 0 \quad \Rightarrow \frac{\partial^2}{\partial x^2}\tilde{c}(t,x) > 0$$

$$(A),\ (B) \Rightarrow \quad \sigma^2(t,x) = \sigma_{ip}^2(t,x)\frac{1 + 2t\dfrac{\partial}{\partial t}\log\sigma_{ip}(t,x)}{1 + x\left(\dfrac{\partial}{\partial x}\log\sigma_{ip}(t,x)\right)\left(\dfrac{t}{2}\sigma_{ip}^2(t,x) - \log(x/S_0)\right)}$$

An example

$$\sigma_{ip}(t,x) = a\,t^{\xi}e^{bx}$$

$$\xi > -\frac{1}{2} \Leftrightarrow (A)$$

$$a^2 > \frac{1}{b\,t^{2\xi+1}S_0}\left(\frac{1}{bS_0}+1\right) \Rightarrow (B)$$

$$\sigma^2(t,x) = \frac{(at^{\xi}e^{bx})^2(2\xi+1)}{1+\frac{bxt^{2\xi+1}}{2}(ae^{bx})^2 - bx\log\frac{x}{S_0}}$$

$$f_t(\log x) = \frac{\phi(\lambda)\left\{1+bx\left(\frac{t}{2}(at^{\xi}e^{bx})^2 - \log\frac{x}{S_0}\right)\right\}}{a\,t^{\xi}\sqrt{t}e^{bx}}$$

A smile is a curve which invites us to a fascinating world.

Von Voyage!